Go and Mathematics

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Around 4,000 years ago in ancient China, a board game developed that would redefine the mathematical study of games. This game is known as 围棋 (Wei Qi), or “Go”. Two players, one playing black and the other white, take turns placing stones on the intersections of a Go board (See Figure 1 and 2). If a stone or group of stones on the board is completely surrounded on all sides, then those stones are “captured” and removed from the game (See Figure 3). At the end of the game, the player with the most territory is the winner. The rules of Go are relatively simple, and anybody could learn them and start playing within 5 minutes. However, there is a deep mathematical complexity in the game that cannot be ignored, and mastering the game requires great dedication and experience, a process that requires a lifetime.

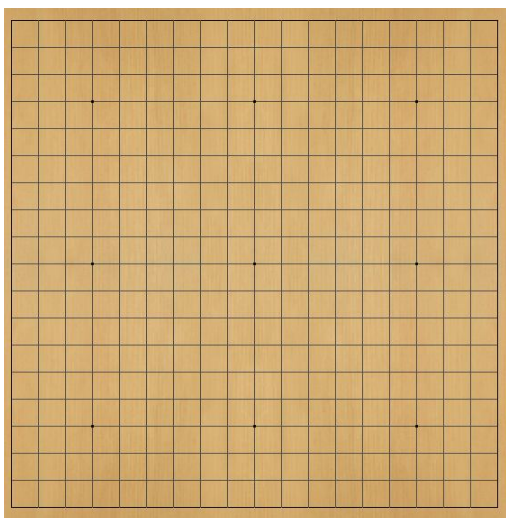


Figure 1: An example of a 19x19 Go Board

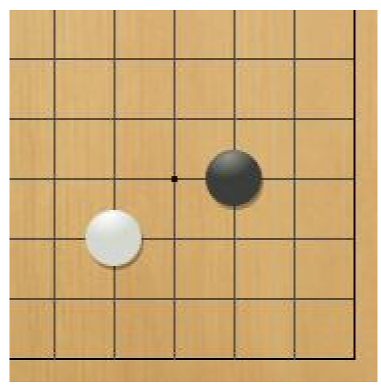


Figure 2: Stones go on intersection points

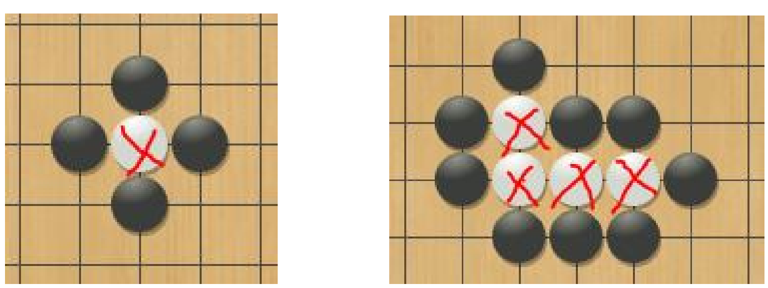


Figure 3: Examples of captured stones (marked with red X’s)

Original Diagrams

Go is an elegant game of balance and strategy. The game is far less about surrounding and capturing than it is about establishing your own territory in an area of the board and placing stones on key points to gain territory. In fact, it is possible for a whole game to play out in which neither side captures a single stone. Stones placed farther apart from each other have the potential of more territory, but also run a higher risk of being separated and killed. Furthermore, in Go it is possible to create territory that is "alive", by creating two or more "eyes" in a group of stones. In Figure 4, the group of white stones that have been surrounded by black cannot be captured. They are considered to be "alive" because they have more than two "eyes" (marked 1 and 2). Black cannot place a stone in location 1 and 2 on the same turn, therefore there is no way for black to completely surround the white pieces.

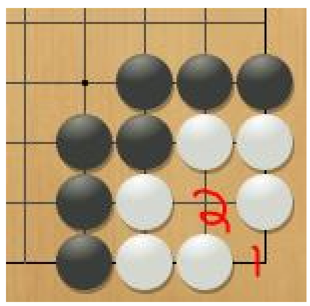


Figure 4: White is alive in this corner of the board

Original Diagram

10^{50}- This is an estimated value based on Shannon's number for the amount of possible legal positions in the game of chess. **10^{80}**- This is the estimated number of atoms in the observable universe. **2×10^{170}**- This unthinkably huge number is the estimated number of possible legal positions on a 19x19 Go board, **without** taking into account the order of moves played. In fact, the number of possible variations in just the first 40 moves of a Go game on a 19x19 board already surpasses the number of atoms in the universe. It is this huge number that makes Go a potential target for mathematical study, as well as differentiates it from other board games in terms of game complexity.

As with most board games, many permutational calculations can be done with Go. The simplest is the number of possible arrangements on the board, assuming that each intersection can have a black stone, white stone, or no stone. This can be expressed as 3^n, with n representing the number of intersections on the board (See Figure 5).

3^n	n	Board Size
4.43×10^{38}	81	9x9
4.30×10^{80}	169	13x13
1.74×10^{172}	361	19x19

Figure 5: The number of possible arrangements for specific board sizes

Original Diagram

On a 19 x 19 Go board with 361 intersection points, the number of possible arrangements is 3^{361}, without taking into consideration the order of moves played. However, this includes many illegal positions, as arrangements in which stones are surrounded on all sides (have no "liberties") do not count. Thus, many Go enthusiasts and mathematicians have tried to calculate the number of legal positions in the game of Go. The number 2×10^{170} came from the calculations of two mathematicians, John Tromp and Gunnar Farnebäck. They noted that as the size of the Go board increases, the percentage of legal positions (out of all possible arrangements) decreases. Using computer programs, they approximated that the percentage of legal positions on a 19 x 19 Go board was close to 1.2%. Using this data, they were able to determine the approximate number of legal positions for the game of Go, which is 1.2% of 3^{361} (See Figure 6). This is a value close to 2.08×10^{170}.

Probability	Illegal positions	Legal positions	Board Size
0.333333	2	1	1x1
0.703704	24	57	2x2
0.643957	7008	12675	3x3
0.564925	18728556	24318165	4x4
0.527724	1646725708	1840058693	4x5
0.235	approximation	approximation	9x9
0.087	approximation	approximation	13x13
0.012	approximation	approximation	19x19

Figure 6: Tromp and Farnebäck's results

http://senseis.xmp.net/?NumberOfPossibleOutcomesOfAGame

Another calculation for Go is in the number of possible games. This is also known as the game's tree complexity, and is different from the number of legal positions of a game in that individual moves are taken into account. The simplest permutation for this would be 361!, which has a value close to 1.44×10^{768}. On the first move, there are 361 intersections to choose from, then 360, then 359, and so on. This calculation is flawed in that it may result in illegal positions and captures and in the simple fact that Go games rarely ever last 361 moves. A calculation by computer scientist Victor Allis uses the fact that typical games of Go will last an average of 150 moves with an average of 250 choices per move, resulting in a value close to 10^{360} ($250^{150} = 4.9 \times 10^{359}$). Currently however, the calculation of Go permutations is generally vague, and many things about the game and its mathematical connection are not clear. Most of the calculations done so far are broad estimates at best (albeit extremely complicated ones using computer programs). As of now, there is a considerably low amount of in-depth mathematical analysis with Go (compared to a game like Chess), simply because of the incapability of even the best of our modern computers to compute such large values. There is also much more that can be done mathematically with Go that is increasingly difficult due to its complexity. For example, given a group of stones in a position on the board, how many different results could be associated with that arrangement? How does the permutational calculation of how one area plays out in a certain number of moves factor into the aspect of the whole game? What about permutations based on capturing stones in an area of the board, or even the occasional occurrence of mutual living (in which both black and white stones cannot be captured and share the space)?

Go's mathematical complexity becomes even more apparent when looking at computer Go programs. Since there are many possibilities for how a game of go can play out, a computer struggles to play a game of Go against even amateur human players. In 1997, a computer chess program known as Deep Blue defeated the world chess champion Garry Kasparov. Computer Go on the other hand, was completely inferior. Go programs such as "HandTalk" challenge 3 Taiwanese inseis to an 11 stone handicap match (See Figure 7) and still lose 1 of the 3 matches.

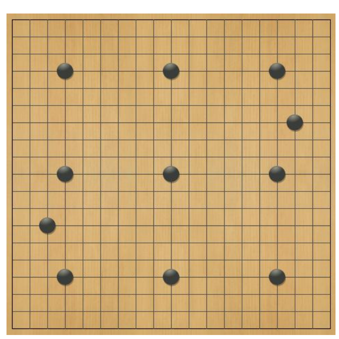


Figure 7: Black (computer) gets an 11 stone "head start" before white plays the first move

Original Diagram

Consider the fact that inseis are only students of Go and that 11 stones is already an extreme advantage in the game (the usual maximum is 9 stones). Then consider the fact that while a Go program still loses 1 of these 3 seemingly unchallenging games, a chess program in the same year has the capability to defeat a chess champion on even grounds. Why is it that computer Go programs struggle to do what computer chess programs can? The reason is behind the variations of the games. In a chess program, the computer looks for as many variations as possible, evaluates the board position, and then propagates back to find the best move. The problem with this method in computer Go is that there are **too many** more variations, making evaluation difficult and counting impossible. However, there has been much improvement since 1997, and this is largely a result of the Monte Carlo Tree Search, a 2006 project that assigned a win rate to a certain move out of a bunch of random plays. To accomplish this, many random plays are played from a certain position until precise scoring can be done, then each of the moves are assigned a win rate based on the results. Using this method, computer Go programs experienced a drastic improvement. For example, a couple of months ago on 06/05/2013, computer Go program “Zen” defeats a strong 9d amateur player with 3 handicap stones. This is still far from being able to defeat a Go champion in an even match, but the programs are constantly being improved and someday they may eventually surpass that of even the best human players.

In my research, I was also interested in the calculation of positions and legal positions on the board. I tried to do my own calculations for the number of positions in a 2x2 corner of the board. Figure 8 on the next page is a tree that shows the number of combinations for the placement of stones turn by turn (with black going first). Black first places a stone in one of 4 starting positions, then each of those 4 starting positions for black are followed up with 3 possible positions for white’s next move, and so on. Afterwards, I looked at each possible final state for the stones, and counted how many of each appeared. Interestingly, there are exactly four ways to reach each of the final stages of the legal stone arrangements listed in Figure 9 (see next page) except for two arrangements: a) and b), which have a total of two ways to reach. All the arrangements take 4 steps to complete (4 intersection points in a 2x2 board) except a), which requires 5 moves to reach and b) which requires 3 moves to reach.

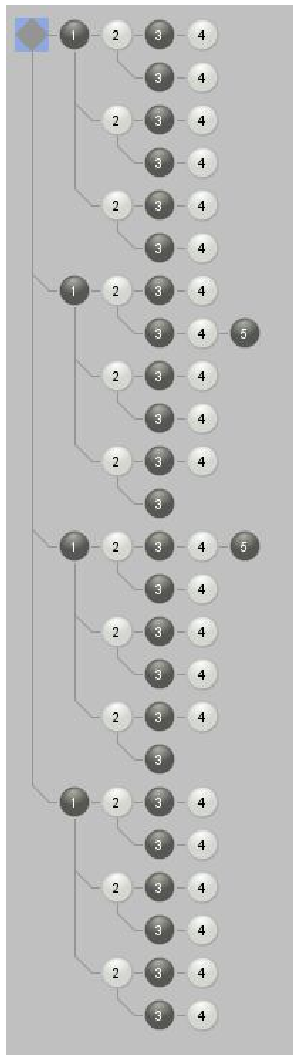


Figure 8: Tree for arrangements on a 2x2 corner of the board.

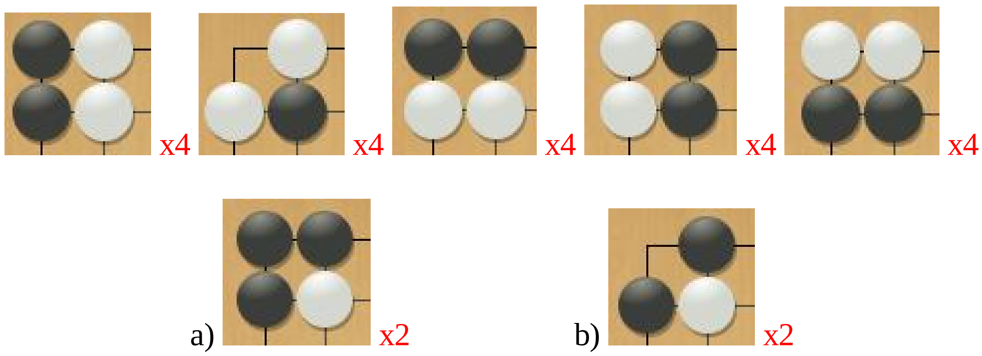


Figure 9: The ending result of arrangements

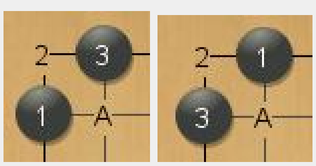
Original Diagrams

1st Move (Black): 4 choices for stone placement

2nd Move (White): 3 choices for stone placement

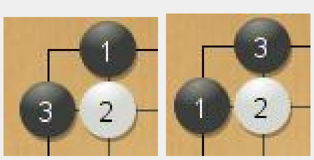
3rd Move (Black): 2 choices for stone placement**

**Capturing will occur in 2 scenarios



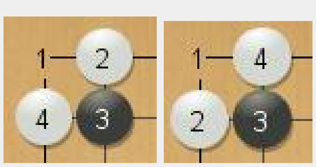
The fourth move for these two scenarios must be on point A

**2 scenarios will not be eligible for a 4th move:



Placing a 4th stone (white's turn) inside the last remaining area is an illegal move.

4th Move (White) 1 choice for stone placement



4 x 3 x 2 x 1 = 24 combinations

Figure 10: Breakdown of each move

Note that Figure 10 above is for the final state of stone arrangements, in which the area is played out completely. Even though there were some limitations for the 3rd move and onwards, this does not affect the end result because it would only be a difference of whether or not you multiply by 1.

Counting up all possible stages of the game (in which the game could potentially stop after as early as the first move) would give you 16 stages for each placement of black in move 1 (4 choices). 4x16 = 64, which is the number of arrangements without a number of moves limitation.

And yet even this is not the full scope of legal positions (when the order of moves is taken into account), because arrangements such as the one shown in Figure 11 are legal but have not been counted above. This is because only arrangements following a tree with black going first in the area were taken into account.

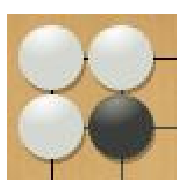


Figure 11: a legal position that can only occur if white went in the area first

Original Diagram

To truly count the number of legal positions in a 2x2 corner, a different approach should be used. First, it can be assumed that each of the 4 positions available hold a black stone, a white stone, or no stone. I chose to represent this with 4^3, or 64 arrangements. 1 arrangement should be subtracted, and that is if all 4 positions have no stones, thus arriving at 63 arrangements. Now, instead of counting the number of legal positions, the number of illegal positions could be counted. This will occur whenever a black stone is surrounded by two white stones or a white stone is surrounded by two black stones. For this to happen, one of the two positions shown in Figure 12 must be present.

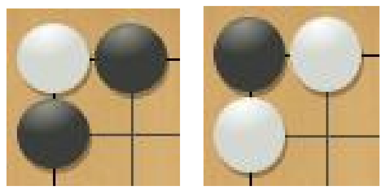
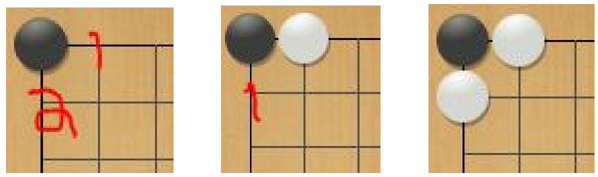


Figure 12: illegal positions

Original Diagram

The question is what happens with the 4th remaining position on the board. For each of the two setups shown in Figure 12, the remaining stone can either be black, white, or empty. This makes 3x2 = 6 illegal positions that after having been subtracted from the previously calculated value of 63, gives 57 legal positions. The method for arriving at this value was different from using 3^n. Instead, I used n^3, for which the resulting amount of illegal positions considered would be different. Even so, this result corresponds with the Tromp and Farnebäck's calculation for the 2x2 corner, although I am not certain it would apply to other sizes for the board.

Going back to my calculations however, I realized that there must have been a problem. If a stone that has no "liberties" (See Figure 12) is on the board, then it is an illegal position. I ignored the fact that in a 2x2 corner, there were boundaries on ALL sides restricting the liberties of stones, not just on the two edges.

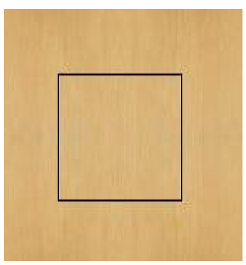


2 liberties 1 liberty 0 liberties

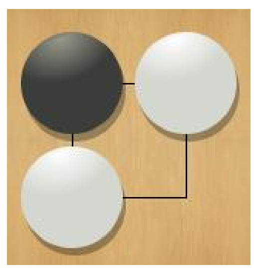
Figure 13: If any stone or group of stones have 0 liberties, then it is an illegal position.

Original Diagram

This also means that if there are no stones without any liberties on the board, the position is legal, and therefore having no stones at all in the area is also considered a legal position. A more restrictive board for 2x2 should be used, the best representation of which would be a square (See Figure 14).



14a.



14b.

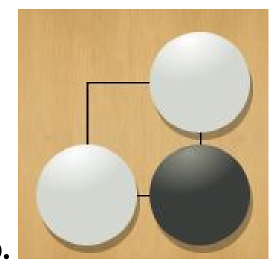


Figure 14: A 2x2 board

Original Diagram

Now, not only would 14a. be an illegal position, 14b. would also be considered illegal. Essentially, my old calculation was of a 2x2 corner of the board without taking into account the fact that the corners and edges of a board count against a stone's liberties. Using this new board, which positions would be illegal? I recognized that any arrangement in which the board was completely filled with stones would be an illegal position. This is because there would be no liberties for any group of stones on the board. To calculate this, I used the fact that each of the 4 positions on the board must have either a black stone or a white stone. This would be equivalent to 2^4, or 16 illegal positions for which the entire board is filled. Furthermore, there is no possible way for an illegal position to be counted if there are only 2 or less stones on the board. This leads me to count all the illegal positions for which there are exactly 3 stones on the board. The only way for this to occur is when a stone in the very corner is surrounded by two stones of the opposite color (See Figure 14a.). Since there are four corners available to place this "dead stone", and two different colors of stones to fill each of the corners, this results in 4x2 = 8 illegal positions for which there are exactly 3 stones on the board. Subtracting 16 illegal positions and 8 illegal positions from the total 81 (3^4) positions results in 57 legal positions.

From this, I can generalize a formula for calculating the number of legal positions, in which *n* represents the number of intersections on the board and I_n represents the number of illegal positions for *n* number of stones on the board:

$$\textit{number of legal positions} = 3^n - (I_n) - (I_{n-1}) - (I_{n-2}) - \square - (I_0)$$

Calculating I_n is different for each value of *n*, and I cannot draw a method for reaching it without individually determining illegal positions. As expected, these numbers grow exponentially in value as the board size grows larger. This experiment with the calculation of legal positions has increased my understanding of the permutational calculations that can be done with Go.

Despite the fact that Go is one of the most played board games in the whole world, it receives fairly little attention mathematically. A very small number of mathematicians and computer programmers contribute to basically all of our current understanding, and progress has been incredibly slow from what we know already. However, as time passes, it is inevitable that more and more of the secrets of Go will be unlocked, with computer go programs becoming more and more powerful in correspondence. Suppose a new method of calculation is discovered or changes to a computer are made in the process of studying Go; It is without doubt that any mathematical advancements in our understanding of Go can also be applied elsewhere. Hopefully, this paper has introduced to you to the mathematics of Go and demonstrated that Go is not just a simple game of surrounding stones.

**Unless mentioned, all figures used in this research paper are original. A program called Fuego was used for most of the original diagrams.

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